DOCUMENT RESUME

ED 245 881 SE 044 574

AUTHOR Thomas, Hoben
TITLE Sex Differences in High Mathematical Reasoning Ability: A Theory to Fit the Facts.
PUB DATE 84
NOTE 22p.
PUB TYPE Reports - Descriptive (141)

EDRS PRICE MF01/PC01 Plus Postage.
DESCRIPTORS *Genetics; *Mathematics Achievement; Mathematics Education; Models; Quantitative Tests; *Scores; Secondary Education; *Secondary School Mathematics; *Sex Differences; *Theories
IDENTIFIERS *Scholastic Aptitude Test

ABSTRACT

In talent searches for gifted junior high school youth, those who score in the top two to five percent on a standardized achievement test are eligible to take the Scholastic Aptitude Test (SAT) which is regarded as a high-level ability test. While no important sex differences on the SAT verbal test have been observed, substantial sex differences in several statistics (including average scores) have consistently favored boys on the SAT mathematics test. A theory is provided which accounts for and provides an explanation of the SAT mathematics sex differences data. The proposed model based on this theory assumes that a recessive X-linked gene facilitates high mathematical score performance. It is pointed out that: (1) predictions from the theory are in precise order agreement with empirical data; (2) new testable predictions have been made; and (3) published data enables bounds on the putative recessive facilitative gene to be given (which may be less than 0.07). It is also pointed out that the model does not "prove" genetic factors are at the root of high mathematical test score sex differences; but until an alternative is proposed and the current model is clearly falsified, it should be regarded as a viable testable explanation. (Author/JN)

****
* Reproductions supplied by EDRS are the best that can be made *
* from the original document. *
****

ED245881

U.S. DEPARTMENT OF EDUCATION
NATIONAL INSTITUTE OF EDUCATION
EDUCATIONAL RESOURCES INFORMATION CENTER (ERIC)
✓ This document has been reproduced as received from the person or organization originating it.
☐ Minor changes have been made to improve reproduction quality.
• Points of view or opinions stated in this document do not necessarily represent official NIE position or policy.

Sex Differences in High Mathematical Reasoning Ability: A Theory to Fit the Facts

Hoben Thomas

The Pennsylvania State University

"PERMISSION TO REPRODUCE THIS MATERIAL HAS BEEN GRANTED BY
Hoben Thomas
TO THE EDUCATIONAL RESOURCES INFORMATION CENTER (ERIC)."

SE044574

Abstract -- In talent searches for gifted junior high school youth those who score in the top 2-5% on a standardized achievement test are eligible to take the Scholastic Aptitude Test (SAT) which is regarded as a a high-level ability test. While no important sex differences on the SAT verbal test have been observed, substantial sex differences in several statistics, including average scores, have consistently favored boys on the SAT mathematics test. A falsifiable theory views gifted mathematical talent as being facilitated by an X-linked recessive gene. Predictions from the theory are in precise order agreement with the empirical data. New testable predictions are also made. Published data enables bounds on the putative recessive facilitative gene to be given. It may be quite small, less than .07.

In two recent Science articles Benbow and Stanley have reported large sex differences on the mathematics portion of the Scholastic Aptitude Test (SAT-M) among extremely able youths. These papers provide an impressive summary of data on talent searches spanning a decade and involving about 65,000 seventh through tenth grade students (1, 2). Since their first paper attracted considerable attention, particularly in Science, it seems likely that their second paper will also (3). Yet to date no theory has appeared which can account for the data even though the major findings have been known for almost a decade (4).

This paper will provide a theory. It will be shown that a single genetically-based sex differences parameter is sufficient to account for and provide an explanation of the SAT-M sex differences data. First, however, a brief review of the talent search methodology and results is necessary.

For the six talent searches prior to 1980 performance in the top 3% (2% to 5% depending on the search) on a standardized mathematics achievement test, such as the School and College Ability Test, was necessary for eligibility to take the SAT-M. Starting in 1980, either a correspondingly high verbal test score or general intellectual test score was acceptable for eligibility (2). Model development will proceed assuming that the selection test was a mathematical achievement test; no important difference in SAT-M statistics appears to have emerged as a result of this change in sampling technique (1, 2). How this eligibility sampling change would alter the model will be noted later.

The facts Benbow and Stanley have reported and for which any theory must account in a coherent way are summarized as (i) - (iii): (i) Sample mean SAT-M scores for boys always exceeded the sample mean for girls. For any search the average difference was at least 30 points (1, 2). (ii) While

not as emphasized, the SAT-M sample variance for boys was larger than the variance for the girls in every search. (iii) For a fixed SAT-M score, s, the ratio of the proportion of boys with scores at or exceeding s, relative to the corresponding proportion of girls, was greater than one; as s increased the ratio increased; e.g., if s = 420 the ratio was 1.5:1, but averaged 13:1 when s = 700 (2). These results contrast sharply with the verbal score data reported for the same subjects. No important sex difference in verbal ability has been found (1, 2).

The proposed model is based on an old idea and assumes a recessive X-linked gene facilitates high mathematical score performance. Years ago Page suggested but did not develop this explanation for the talent search sex differences data and it has been offered by others as an explanation for certain numerical skill and spatial task sex differences (5). In recent years however, the X-linked gene hypothesis has fallen intc disfavor. A main reason is that there has been a failure to embed X-linked facilitated performance into a model with clear implications for data. Consequently, certain family data, which have often been viewed as providing evidence against the X-linked hypothesis, have been found, upon closer inspection, to provide no test of the hypothesis at all (6). Furthermore, recent tests of the X-linked hypothesis using different techniques have provided support for it (7, 8).

To develop the model assume for the moment that there is an absence of all test score influences except for the X-linked genetic influence. Let the X-linked gene have two alleles, but only the recessive form with frequency q, $0 < q < 1$, has facilitative effects. Assume for the moment, that all individuals whose performance is facilitated by the X-linked effect have identical point-specific high score outcomes x_2 and y_2 on <u>two</u> tests; x_2 may be regarded as an achievement score and y_2 an SAT-M score.

Correspondingly, those without the facilitative genetic effect have point-specific outcomes x_1 and y_1 where $x_1 < x_2$, and $y_1 < y_2$. Then under a simple genetic model the frequency of the pair (x_2, y_2) is q in boys and q^2 in girls, while the pair (x_1, y_1) has frequency $1 - q$ for boys and $1 - q^2$ for girls (6). To summarize formally define a random variable B with mass function

$$b(x_i, y_i) = \begin{cases} 1 - \lambda & \text{if } i = 1 \\ \lambda & \text{if } i = 2 \\ 0 & \text{otherwise,} \end{cases}$$

where $x_1 < x_2$, $y_1 < y_2$ and $0 < \lambda < 1$. For boys $\lambda = q$, $1 - \lambda = 1 - q$, while for girls $\lambda = q^2$ and $1 - \lambda = 1 - q^2$. An example of $b(x_i, y_i)$ is given in Figure 1a.

Of course people do not obtain point-specific test outcomes; a continuum of scores is always observed. B does however model the X-linked influence. Assume all other test related influences such as socialization effects and measurement error are summarized in a bivariate normal random variable N independent of B. Let N have the familiar density function $f(x, y)$, $-\infty < x, y < \infty$, with means μ_x, μ_y, variances $\sigma_x^2 > 0$, $\sigma_y^2 > 0$ and ρ, $-1 < \rho < 1$ (9, p. 288). Now add these different sources of influence by adding B and N. The distribution of B + N is called a bivariate normal mixture distribution with function

$$u(x, y) = (1 - \lambda)f_1(x, y) + \lambda f_2(x, y) \qquad (1)$$

where the $f_i(x,y)$ components are bivariate normal densities with component parameters identical to $f(x, y)$ except that $\mu_{x1} = \mu_x + x_1$, $\mu_{x2} = \mu_x + x_2$,

$\mu_{y1} = \mu_y + y_1$, $\mu_{y2} = \mu_y + y_2$; thus $\mu_{x2} > \mu_{x1}$ and $\mu_{y2} > \mu_{y1}$ (10). Since test scores are always positive, it will be assumed throughout that all μ parameters are positive.

Eq. 1 is the basic model density structure, where X is to be regarded as the achievement test score variable, and Y the SAT-M variable throughout. To assist intuition, an example of $u(x, y)$ is shown in Figure 1d. Note that ρ is not the correlation of X and Y in Eq. 1; even if $\rho = 0$ X and Y may be highly correlated (11).

Straight-forward integration of Eq. 1 gives the marginal distributions which are identical in form:

$$v(x) = (1 - \lambda)g_1(x) + \lambda g_2(x) \tag{2}$$

and

$$w(y) = (1 - \lambda)h_1(y) + \lambda h_2(y), \tag{3}$$

where $v(x)$ and $w(y)$ are two-component normal mixture densities; g_i and h_i are univariate normal densities, with parameters μ_{xi}, σ_x^2, μ_{yi}, σ_y^2, $\mu_{x2} > \mu_{x1}$, and $\mu_{y2} > \mu_{y1}$. An example of $w(y)$ is pictured in Figure 1b. For certain model parameter values, $w(y)$ may be bimodal, although typically not for the values of the parameters considered here (cf. 9, p. 32).

From Eq. 3, the SAT-M distribution for boys is

$$w_b(y) = (1 - q)h_1(y) + qh_2(y) \tag{4}$$

and for girls

$$w_g(y) = (1 - q^2)h_1(y) + q^2 h_2(y) \qquad (5)$$

where the subscripts b and g denote boys and girls.

It is proposed that Eqs. 4 and 5 be regarded as candidates for modeling the talent search SAT-M score distributions if all persons tested on the achievement test were also tested on the SAT-M. Since the talent search procedure specifies that only the top 3% or so of achievement test scorers are eligible to take the SAT-M, the model does not strictly apply to the talent search situation. However, the model that <u>does</u> apply is simply a special case of this model where the distribution of X is truncated as some upper point t such that P(X > t) is about 3%. It is relevant to consider the properties of Eqs. 4 and 5 as they relate to point (i) - (iii) above since these results will anticipate similar results in the truncated case. The means and variances of Eqs. 4 and 5 are easily obtained; they also appear elsewhere (7). The mean of the boys is

$$\mu_b = (1 - q)\mu_{y1} + q\mu_{y2},$$

and for the girls the mean is

$$\mu_g = (1 - q^2)\mu_{y1} + q^2\mu_{y2}.$$

It is immediate that $\mu_b > \mu_g$, i.e., the mean of the boys must always exceed the mean of the girls independent of other model parameters, a result in agreement with sample data, i.e., point (i) above.

It is worth pausing to stress why this result occurs since it also provides an explanation of the sample mean difference. It is precisely because

high scoring is assumed to be recessively facilitated and because $q > q^2$; thus there are <u>proportionally</u> more high scoring boys than girls.

The variances of Eqs. 4 and 5 are

$$\sigma_b^2 = \sigma_y^2 + (\mu_{y2} - \mu_{y1})^2 q(1 - q)$$

and

$$\sigma_g^2 = \sigma_y^2 + (\mu_{y2} - \mu_{y1})^2 q^2(1 - q^2) \ .$$

It follows easily that

$$\sigma_b^2 > \sigma_g^2 \text{ iff } q < (5^{1/2} - 1)/2 = .618. \qquad (6)$$

Thus the variance of the boys exceeds the variance of the girls if $q < .618$, and under this condition the variance ordering corresponds with the sample data point (ii) above. Note that the model would be falsified at this point if the sample variance for boys were less than the sample variance for girls because under the model this would imply $q > .618$. Clearly superior mathematical talent is a rare trait and thus q must be small. Later, empirical facts will suggest q is much smaller than is forced by Expression 6.

To show the model is in agreement with (iii) above define from Eqs. 4 and 5

$$r(s) = [\int_s^\infty w_b(y)dy] \ / \ [\int_s^\infty w_g(y) \ dy] \ , \ -\infty < s < \infty. \qquad (7)$$

which is a ratio of the proportion of boys with SAT-M scores above score s, to the corresponding proportion of girls. Since r(s) is always greater than one and increases as s increases, the model agrees with the empirical facts, point (iii) above (12).

A testable implication of the above development is that the distribution of mathematical achievement test scores, such as test standardization data, for boys and girls separately should be a two-component normal mixture distribution specified by Eq. 2. As Figure 1b makes clear however, such distributions may appear very normal in some data sets. However, most test results such as standardization data are based on combined samples of boys and girls. This combined sample is also from a mixture distribution. For instance, if there are equal numbers of boys and girls tested, as in standardization samples, then the distribution would be, from Eq. 2,

$$\tfrac{1}{2}\,(v_b(x) + v_g(x)) = (1 - q_*)g_1(x) + q_* g_2(x) \qquad (8)$$

where $q_* = (q + q^2)/2$. If $q = .2$, then $q_* = .12$; .12 and 1 - .12 = .88 are the parameters used to generate the functions in Figure 1.

A consequence of this interpretation provides an explanation for another not understood empirical fact: The proportion of eligible boys has always exceeded the proportion of eligible girls in every talent search (1, 2). From Eq. 8 the total proportion of eligible youth, as determined by standardized achievement test norms, would be

$$\tfrac{1}{2}\int_t^\infty v_b(x)dx \quad + \quad \tfrac{1}{2}\int_t^\infty v_g(x)dx,$$

where t is the achievement score critical value. Since the first integral is always larger than the second (recall that, similarly, $r(s) > 1$, Eq. 7) under the model more boys will be eligible to take the SAT-M than girls. In the searches prior to 1980 57% of the entries were boys and 43% were girls (1). Since the new 1980 eligibility criteria include non-mathematical

achievement tests including high verbal test scores to which the proposed model does not apply, it might be expected that the overall proportion of girls selected would be more similar to the proportion of boys selected, which is precisely what the recent data suggest (2).

The properties of the SAT-M test score distribution, Eqs. 4 and 5, are in excellent correspondence with the empirical facts points (i) - (iii) above. However, this general model does not reflect the talent search sampling procedure. To change the model to accurately reflect the talent search selection scheme means restricting attention to the 3% of X scores above the cutting or truncation point, t. The joint (x,y) space would then be $t \leq x < \infty, -\infty < y < \infty$. This apparently minor change greatly complicates the problem analytically largely because all expressions of interest include normal integrals which cannot be written in closed form. Fortunately however, most of the properties of interest concern the SAT-M distribution $w_t(y)$; these properties may be obtained by straight-forward, although often tedious integration.

The alteration begins by rewriting Eq. 1 as

$$u_t(x, y) = (1 - \lambda)f_{t1}(x, y) + \lambda f_{t2}(x, y), \quad t \leq x < \infty, -\infty < y < \infty. \quad (9)$$

The $f_{ti}(x, y)$ are normal bivariate components truncated at t. An example of $u_t(x, y)$ is illustrated in Figure 1e. Integration of Eq. 9 gives $w_t(y)$, and similarly the first two moments of Y (13). An example of $w_t(y)$ is shown in Figure 1c. A major difference of the resulting expressions from those given above is that the properties of the SAT-M distribution depend on parameters of the achievement test distribution.

It is straight forward to show the ordering of the boys' mean, μ_{tb} and the girls' mean, μ_{tg} is specified by the following condition:

$$\mu_{tb} > \mu_{tg} \text{ iff } (\mu_{y2} - \mu_{y1})/\sigma_y > \rho(R_1^{-1} - R_2^{-1}) \qquad (10)$$

where $R_i^{-1} = [\phi((t - \mu_{xi})/\sigma_x)] / [1 - \Phi((t - \mu_{xi})/\sigma_x)]$; $\phi(\cdot)$ is the standard normal density, and $\Phi(\cdot)$ is the standard normal cumulative lower tail. R_i, called Mill's ratio, is the ratio of upper tail to the density.

Because (see 14)

$$(t - \mu_{x1})/\sigma_x - (t - \mu_{x2})/\sigma_x = (\mu_{x2} - \mu_{x1})/\sigma_x > R_1^{-1} - R_2^{-1}, \qquad (11)$$

a more intuitive sufficient condition is that

$$\mu_{tb} > \mu_{tg} \text{ if } (\mu_{y2} - \mu_{y1}) / \sigma_y > \rho\left[(\mu_{x2} - \mu_{x1}) / \sigma_x\right]. \qquad (12)$$

Again, these conditions may be viewed as ways of interpreting the orderings of the observed sample means, or as stating conditions under which the model holds.

In Expressions 10 and 12 if $\rho = 0$ or $t = -\infty$, (i.e., there is no truncation) then $\mu_{tb} > \mu_{tg}$ always holds. Since ρ reflects test covariation within each bivariate component and is not the X and Y correlation, ρ might not necessarily be large since X and Y may be correlated even when $\rho = 0$ (15).

The difference $R_1^{-1} - R_2^{-1}$ increases as $(t - \mu_{x2})/\sigma_x$ increases (i.e., as t increases) but decreases as $(t - \mu_{x1})/\sigma_x$ increases (16). Consequently, the difference $R_1^{-1} - R_2^{-1}$ becomes larger as t increases. It is maximum at $t = \infty$ when $R_1^{-1} - R_2^{-1} = (\mu_{x2} - \mu_{x1})/\sigma_x$. In practical cases the differences between the right and left hand terms of Expression 11 may be relatively large. For example in Figure 1, $(\mu_{x1} - \mu_{x2})/\sigma_x = 2$, $R_1^{-1} - R_2^{-1} = 1.70$.

Next, consider where t the truncation point might be located relative to the component means, μ_{x1} and μ_{x2}.

Recall that eligible achievers must score in the upper 3% or so of the achievement test score distribution; this tail region is approximately the area above a point corresponding to the mean plus two standard deviations for most any distribution, and more specifically for the mixture distribution as well.

This fact forces the following result: if $\lambda > .2$, it must be that a point corresponding to the mean plus two standard deviations (and thus t) must be above μ_{x2}. Even if $\lambda < .2$ it is quite likely $t > \mu_{x2}$ if σ_x^2 is not negligible (17). Thus for boys any $q > .2$, and for girls any $q^2 > .2$ means $t > \mu_{x2}$.

The location of t is related to a more intriguing question regarding the possible size of q the frequency of the facilitative allele. While a point-estimate of q would require the actual data, simply knowing that the pre-1980 eligibility ratio of boys to girls was 57/43 (1) and that the combined eligible proportion was .03 suggests q is quite small. For instance, if $t = \mu_{x2}$ then $q < .07$ (18). It is interesting that the gene frequency suggested by the talent search data is much smaller than estimates of .5, 2/3, and .9 which have been suggested for other cognitive tasks where X-linked mediation has been hypothesized (7, 8, 19). Thus, it may not be unreasonable to suggest that there is something unique about rare mathematical talent.

The SAT-M variance expressions are too complicated for a general result to be given concerning their orderings in the truncated case. But an exact result can be given when $\rho = 0$, and speculation can be provided regarding the form of the equivalency when $\rho > 0$.

Consider

$$\sigma^2_{tb} > \sigma^2_{tg} \quad \text{iff} \quad j(\rho)z(t) < k(q), \tag{13}$$

where $z(t) = [1-\Phi((t-\mu_{x2})/\sigma_x)]/[1-\Phi((t-\mu_{x1})/\sigma_x)]$

and where k(q) depends only on q. When $\rho = 0$, $j(0) = 1$, and Expression 13 is exact, so consider this case first.

Now $z(t) > 1$ for all real t, and increases with t; but k(q) increases with decreasing q. Consequently, the larger t is, the smaller q must be for Expression 13 to hold (20). Thus, the variance ordering in the truncated case also sharply constrains the size of q. For instance, with the component means, μ_{x1} and μ_{x2} separated by σ_x and $t = \mu_{x2}$ $q < .38$ which again suggests q must be small. For the truncated case with $\rho > 0$, numerical work suggests that k(q) need not exceed z(t) for the boy's variance to exceed the girl's variance. Thus it is speculated that the appropriate condition for ordering the variances is Expression 13 with $0 < j(\rho) \leq 1$, $j(\rho)$ decreasing with $\rho \geq 0$ increasing.

Finally, it may be simply stated that the result given for Eq. 7, namely that the ratio of proportion of boys relative to girls with SAT-M scores above any score s exceeds one and increases as s increases, holds without qualification for the truncated case as well (21).

This development has shown that derivations from a falsifiable model with one sex difference parameter are in correspondence with all the ordered data relations (i) - (iii) of Benbow and Stanley (1, 2). Furthermore, the constraints imposed on the model by empirical data, particularly that q must be small, appear reasonable and necessary for the model to be plausible. The model also makes testable predictions: particularly that mathematical

achievement test scores (untruncated) should be a two-component mixture distribution. It is unfortunate that the conceptual complexity of the model is increased with the addition of the truncation parameter t; yet it is difficult to imagine a simpler model accounting for the data as well. At least six different parameters seem to be required to account for the ordered data relations (i) - (iii): q, t, ρ, since it could have been assumed that $\mu_{x1} = \mu_{y1} < \mu_{x2} = \mu_{y2}$, $\sigma_x = \sigma_y$. Since ρ is not the model correlation, if $\rho = 0$ (admittedly implausible) the model is still in agreement with the data (and considerably simplified in the truncated case) and there are five unknown parameters. Any other model would seem to require at least as many: different sex SAT-M mean and variance parameters, a correlation and truncation parameter yielding six, and perhaps without any first-principle basis as in the proposed model for why the between sex difference parameters were caused to be different in the first place.

The problem of parameter estimation and model fitting would require the achievement test and SAT-M data pairs (22); in the truncated model some simplifying assumptions would be necessary for the approach to be feasible. In the untruncated case, standard procedures are available, at least for the univariate features of the model (23).

No model of reality may be regarded as correct. A clear point of model vulnerability concerns the assumed bivariate normality of N. No psychological test scores are normal (nor a mixture of normal distributions) at least not exactly, since test scores are positive. Furthermore, it is in the tails of the normal distribution where the discrepancy between reality and theory is likely to be greatest, and it is precisely the tail areas that provide fascinations for the talent search data. Note however, that the model predictions concerning the ordering of the means and variances

in the untruncated case are not dependent on normality assumptions (other error distributions might do as well); most remaining developments were critically dependent on normality.

Since the model does not apply to boys and girls selected by verbal or other non-mathematical tests if the number so selected by such tests were not negligible, the model would need to be altered. For instance, Eq. 1 would require three components: the third representing the proportion selected by the verbal test. Such a model, with truncation, would be intractible. There is clearly a simple alternative: Consider data for those selected by different tests separately. Benbow and Stanley make no comment regarding the proportions selected with different types of tests (2).

In no way does the model "prove" genetic factors are at the root of high mathematical test score sex differences; science does not work that way. But until an alternative is proposed and the current model is clearly falsified, it should be regarded as a viable testable explanation.

One empirical fact that may not be in agreement with the model is that in at least each talent search prior to 1980 a boy has received the highest SAT-M score (1). The model predicts only that the high scoring girls are less probable. A persistence of this finding may force model changes. Perhaps such a different reflects a male environmental advantage.

References and Notes

1. C. Benbow and J. Stanley, Science 210, 1262 (1980).
2. C. Benbow and J. Stanley, Science 210, 1029 (1983).
3. There were eight letters in Science 212, 114 (1981); news articles in Science 210, 1234 (1980), and Science 222, 1312 (1983).
4. D. P. Keating in Intellectual talent: Research and development, D. P. Keating, Ed. (Johns Hopkins, Baltimore, 1976), pp. 23-31.
5. E. Page in Intellectual talent: Research and development, D. P. Keating, Ed. (Johns Hopkins, Baltimore, 1976), pp. 295-307; D. Garron, Psy. Rev., 77, 147 (1970).
6. H. Thomas, Psy. Bull. 93, 427 (1983).
7. H. Thomas, J. Math. Psy. 26, 169 (1982).
8. H. Thomas and W. Jamison, Dev. Rev. 1, 274 (1981).
9. J. Patel and C. Read, Handbook of the normal distribution (Dekker, New York, 1982).
10. The distribution of B + N can be obtained by factoring the product of the characteristic functions then applying a lemma of W. Feller An introduction to probability theory and its applications, Vol. II, 2nd, (Wiley, New York, 1971), p. 504; see also similar proof in (7). The distribution may also be obtained conditionally.
11. The correlation of X and Y in Eq. 1 is

$$\frac{\rho\sigma_x\sigma_y + \lambda(1-\lambda)(\mu_{x2}-\mu_{x1})(\mu_{y2}-\mu_{y1})}{[(\lambda\ (1-\lambda)\ (\mu_{x2}-\mu_{x1})^2 + \sigma_x^2)\ (\lambda\ (1-\lambda)\ (\mu_{y2}-\mu_{y1})^2 + \sigma_y^2)]^{\frac{1}{2}}}$$

12. An easy consequence of the definition shows r(s) is always greater than one. To show r(s) is increasing in s, differentiate r(s) with respect to s, apply result in (14) showing the difference between two Mill's ratios is positive, which shows r'(s) is positive.

13. $$w_t(y) = c\sum_{i=1}^{2} \lambda_i \left[1 - \Phi\left\{\frac{t-\mu_{xi}-\rho(\sigma_x/\sigma_y)(y-\mu_{yi})}{\sigma_x\sqrt{1-\rho^2}}\right\}\right]\sigma_y^{-1}\phi\left(\frac{y-\mu_{yi}}{\sigma_y}\right)$$

where $\Phi(\ .\ )$ and $\phi(\ .\ )$ are the standard normal lower tail and density functions respectively, and

$$1/c = \sum_{i=1}^{2} \lambda_i \left[1 - \Phi\left((t-\mu_{xi})/\sigma_x\right)\right];\quad \lambda_1 = 1-\lambda,\ \lambda_2 = \lambda.$$

$$E(Y) = c\sum_{i=1}^{2} \lambda_i \left\{\left[1 - \Phi\left((t-\mu_{xi})/\sigma_x\right)\right] + \rho\sigma_y\,\phi\left(\frac{t-\mu_{xi}}{\sigma_x}\right)\right\}$$

$$E(Y^2) = c\sum_{i=1}^{2} \lambda_i \left\{\phi\left(\frac{t-\mu_{xi}}{\sigma_x}\right)\rho\sigma_y\left[\rho\sigma_y\left((t-\mu_{xi})/\sigma_x\right) + 2\mu_{yi}\right]\right.$$

$$\left. + \left[\mu_{yi}^2 + \sigma_y^2\right]\left(1 - \Phi\left(\frac{t-\mu_{xi}}{\sigma_x}\right)\right)\right\}.$$

14. Proof sketch: Let Z be a standard normal variable truncated from below at z; $E(Z) = \phi(z)/(1-\Phi(z)) = m(z) = R^{-1}(z) > 0$ for all z. By examining $m'(z)$ and using known bounds on $R(z)$, Mill's ratio, see (9), (3.7.1), p. 64, $m'(z) > 0$ for all z. Thus $R^{-1}(z)$ increases in z; note that $R^{-1}(z_1) - R^{-1}(z_2) > 0$ if $z_1 > z_2$.

Define $n(z_1,z_2) = (R^{-1}(z_1) - R^{-1}(z_2))/(z_1-z_2)$, where $z_1-z_2 > 0$. Clearly $z_1 - z_2 > R^{-1}(z_1) - R^{-1}(z_2)$ iff $n(z_1,z_2) < 1$; certainly $n(z_1,z_2) < R^{-1}(z_1)/(z_1-z_2)$. This last inequality can be bounded from above at 1 using the same Mill's ratio bound and passing to the limit, therefore $z_1 - z_2 > R^{-1}(z_1) - R^{-1}(z_2)$ with $z_1 > z_2$, $z_1 > 0$.

15. The correlation cannot be given. For parameters of Figure 1, but with $\rho = 0$ numerical work suggests that the correlation is about .4.

16. Proof sketch: Let $z_i = (t-\mu_{xi})/\sigma_x$, then $z_1 > z_2$ since $\mu_{x2} > \mu_{x1}$. Define $n(z) = z - R^{-1}(z)$. Using bounds on $R(z)$ in (9) (3.7.5) paragraph (b), p. 65, $n'(z) > 0$, for all z; thus n(z) increases in z. Therefore $n(z_1) > n(z_2)$ is equivalent to Expression 11, and as $z_1 - z_2$ increases, $R^{-1}(z_1) - R^{-1}(z_2)$ increases.

17. $E(X) = (1-\lambda)\mu_{xi} + \lambda\mu_{x2}$; $Var(X) = \sigma^2 = \sigma_x^2 + \lambda(1-\lambda)(\mu_{x2} - \mu_{x1})^2$. $t \doteq E(X) + 2\sigma > \mu_{x2}$ iff $4 > ((\mu_{x2}-\mu_{x1})/\sigma_x)^2 \quad [1-6\lambda+5\lambda^2]$. The quantity $[\cdot]$ is negative for any $\lambda > .2$, and thus $t > \mu_{x2}$ for all $\lambda > .2$.

18. $.03 = \int_t^\infty \tfrac{1}{2}(v_b(x) + v_g(x))dx$; $43/57 = (\int_t^\infty v_g(x)dx)/(\int_t^\infty v_b(x)dx)$ which imply $.034 = \int_t^\infty v_b(x)dx = (1-q)(1-\Phi((t-\mu_{x1})/\sigma_x)) + q[1-\Phi((t-\mu_{x2})/\sigma_x)] > q(1-\Phi((t-\mu_{x2})/\sigma_x))$. For example, if $t = \mu_{x2}$, $.034 > q(1-\Phi(0)) = .5q$, or $q < .068$.

19. R. Bock and D. Kolakowski, Am. J. Hum. Genetics 25, 1 (1973).

20. Expression 13, $j(0) = 1$, follows directly from the variance formula from (13), with
$k(q) = ((1-q^2)-(1-q)(q(1+q))^{\frac{1}{2}})/(q(q(1+q))^{\frac{1}{2}}-q^2)$.
Straight forward calculations show $k'(q) < 0$, $z'(t) > 0$.

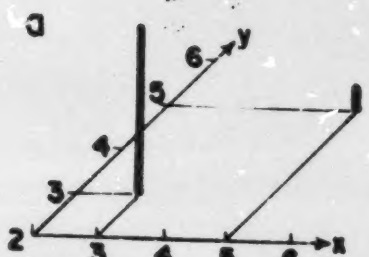
21. Proof sketch: Define r(s) as in Eq. 7 from $w_t(y)$ in (13), and examine r'(s); $r'(s) > 0$ is equivalent, after simplification, to the difference between two integral functions, say $d(p_1) - d(p_2)$ being positive. Under the model $p_1 > p_2$ (which are functions of parameters), and since $d'(p) > 0$, it follows that $r'(s) > 0$.

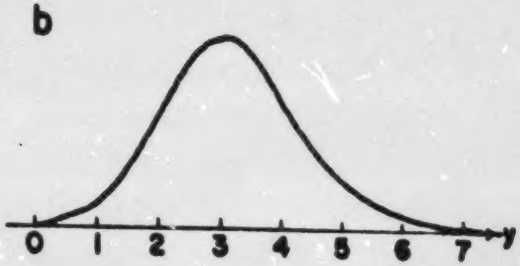
22. June 1983 correspondence with C. Benbow suggests the Johns Hopkins data are not readily available.

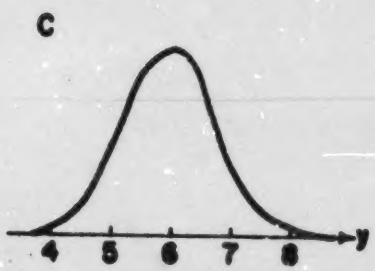
23. A readable paper is M. Aitken and G. Wilson, Technometrics 22, 325 (1980); an example is in (7). The problem of parameter estimation in a bivariate setting is discussed briefly in (6).

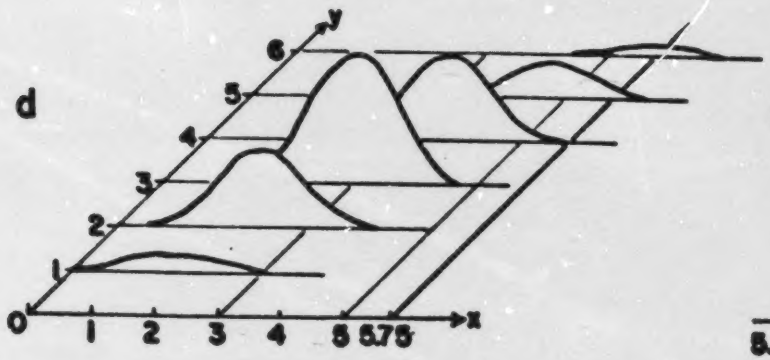
Figure Caption

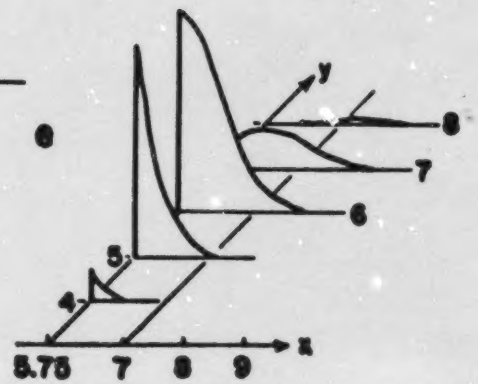
Figure 1. Pictorial representation of distributional components of model. Ordinate scale varies. (a) Distribution of $b(3,3) = .88$, $b(5,5) = .12$; (b) Marginal distribution of x or y for bivariate distribution in panel d; (c) Marginal distribution of y for truncated bivariate distribution in panel e; (d) Two component bivariate normal mixture distribution with component means 3 and 5, and unit variances for both x and y; $\rho = .75$ for both components. Weights are .88 for component centered at (3,3) and .12 for component centered at (5,5); (e) Resulting bivariate distribution when the distribution in panel d is truncated at $t = x = 5.75$. This truncation point leaves .03 in upper tail of the x distribution, panel b; (f) Marginal distribution of x for truncated distribution in panel e.

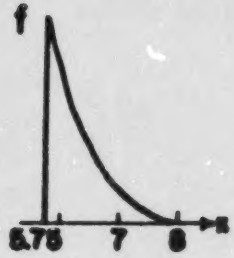

a
y
6
5
4
3
2
3
4
5
6
x


b
0
1
2
3
4
5
6
7
y


c
4
5
6
7
8
y


d
y
6
5
4
3
2
1
0
1
2
3
4
5
5.75
x


e
y
8
7
6
5
4
5.75
7
8
9
x


f
5.75
7
8
x